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INFINITARY RELATIONS

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RESUME :

Relations infinitaires

Notons A^{∞} l'ensemble des mots finis ou infinis sur un alphabet fini A, et $(A^{\infty})^{(k)}$ le produit cartésien de k copies de A^{∞}. Une relation infinitaire est par définition une partie de $(A^{\infty})^{(k)}$. La motivation essentielle de cette étude est que l'ensemble des comportements d'un système composé de k processus réalisant des actions de A peut naturellement être défini comme une relation infinitaire. Nous démontrons des résultats analogues à ceux qui valent dans le cas des langages infinitaires (parties de A^{∞} ou relations de dimension $k = 1$). La situation est cependant plus compliquée, ceci venant du fait qu'un multimot $\vec{\alpha}$ de $(A^{\infty})^{(k)}$ peut avoir des composantes finies et des composantes infinies. Intuitivement c'est ainsi qu'on peut modéliser le comportement d'un système dans lequel certains processus composants sont indéfiniment activés alors que d'autres s'arrêtent pour toujours.

Un prochain rapport traitera du cas important des relations infinitaires rationnelles.

ABSTRACT :

Infinitary relations

If A^{∞} denotes the set of all finite and infinite words on a finite alphabet A, we call infinitary relation any subset of the cartesian product $(A^{\infty})^{(k)}$ of k copies of A^{∞}. The set of possible behaviours of a system composed of k processes whose alphabet of actions is contained in A can be viewed as such an infinitary relation, and this is the main motivation to the present work. We extend results obtained in the case of infinitary languages, i.e. subsets of A^{∞} or relations of dimension $k = 1$. The main difficulty is that a multiword $\vec{\alpha}$ in $(A^{\infty})^{(k)}$ may have finite and infinite components, this corresponding intuitively to the fact that in a system of processes some may be activated for ever when some other come after a while to a complete stop.

A second report will soon follow devoted to infinitary rational relations.

I - INTRODUCTION

If one considers the set of finite behaviours of a process p as a subset of the free monoid generated by the finite alphabet of actions A, let us denote it $HR^{\star}(p)$, one is lead to extend it to infinity to include infinite behaviours which are infinite words on the alphabet A. The set $HR^{\omega}(p)$ of such infinite behaviours is, in the normal case of a process p which has the finite non determinism property, linked to $HR^{\star}(p)$ by the formula :

$$HR^{\omega}(p) = \{u \in A^{\omega} \mid \forall n \in \mathbb{N} \quad u[n] \in HR^{\star}(p)\}$$

In other words an infinite sequence of actions $u \in A^{\omega}$ is an infinite behaviour of p if and only if the initial segment of length n of u (denoted u[n]) is a finite behaviour of p for all n.

And we can reformulate this by writing simply :

$$HR^{\omega}(p) = \mathrm{Adh}\ (HR^{\star}(p))$$

by using the notion of adherence as it is defined in [3]. The adherence of a finitary language $L \subseteq A^{\star}$ is the set of cluster points of L in the natural metric topology on $A^{\infty} = A^{\star} \cup A^{\omega}$.

Now if a finite number k of processes $p_1,\ldots,p_k$ behave simultaneously respecting some synchronisation condition S, a finite behaviour of the system $(\vec{p}, S)$ thus formed is a k-uple of words $<f_1,\ldots,f_k>$ where for all $i \in [k]$, $f_i \in HR^{\star}(p_i)$.

In [7] it is suggested that a general form of a synchronisation condition is the pair $<p_S, \vec{\varphi}>$ of a synchronizing process p_S and a multimorphism $\vec{\varphi} = <\varphi_1,\ldots,\varphi_k>$ where for all $i \in [k]$ φ_i maps the alphabet A of p_S into $A_i \cup \{\varepsilon\}$ where A_i is the alphabet of actions of p_i.

Then $HR^{\star}(\vec{p}, S)$ is exactly the set of all k-uples $<f_1,\ldots,f_k>$ which lie in $HR^{\star}(p_1) \times \ldots \times HR^{\star}(p_k)$ and satisfy the synchronisation condition :

$$\exists\, g \in HR^{\star}(p_S) \quad : \forall\, i \in [k] \quad \varphi_i(g) = f_i$$

We write :

$$HR^{\star}(\vec{p},S) = (HR^{\star}(p_1) \times \dots \times HR^{\star}(p_k)) \cap \vec{\varphi}\,(HR^{\star}(p_S))$$

A question is immediately raised which is to define the infinite behaviours of the system $(\vec{p},\ S)$. Given a finitary relation $R \subset A_1^{\star} \times \dots \times A_k^{\star}$ can we define such a thing as the adherence of R and write $HR^{\omega}(\vec{p},S) = \mathrm{Adh}(HR^{\star}(\vec{p},S))$?

It is immediate to see that this question cannot be answered as easily in the case of relations as in the case of languages though it is of the utmost importance for the study of systems of synchronized processes. The present paper is devoted to the mathematical problem of extending to infinity finitary relations : it should be read as a sequel of [6] and a companion paper to [7] . It is an essential part of a theory of infinite computations and synchronization which the author tries to build in close collaboration with André Arnold [1].

II - LEFT FACTORS AND ENTENDABILITY

For words in A^∞ the notion of left factors is well-known. We define it in the following way :

- if $f = f(1) \dots f(n)$ is a word of length n in A^* and $p \in \mathbb{N}$
  $f[o] = \varepsilon$ where ε is the empty word
  $f[p] = f(1) \dots f(p)$ for $1 \leq p \leq n$
  $f[p] = f[n] = f$ for $n \leq p$

- if $u = u(1)\ u(2) \dots u(n) \dots$ is an infinite word in A^ω and $p \in \mathbb{N}$
  $u[o] = \varepsilon$
  $u[p] = u(1) \dots u(p)$ for $p \geq 1$

Then for all $\alpha \in A^\infty$ we define

$$FG(\alpha) = \{\alpha[p] \mid p \in \mathbb{N}\} \text{ which is a subset of } A^*$$

The product on A^∞ is defined by :

- if $f \in A^n$ and $g \in A^p$, fg is the word of length $n+p$ given by :
  $(fg)(\ell) = f(\ell)$ for $\ell \leq n$
  $(fg)(\ell) = g\ (\ell-n)$ for $n+1 \leq \ell \leq n+p$

- if $f \in A^n$ and $u \in A^\omega$, fu is the infinite word given by
  $(fu)(\ell) = f(\ell)$ for $\ell \leq n$
  $(fu)(\ell) = u(\ell-n)$ for $\ell \geq n+1$

- if $u \in A^\omega$ and $\alpha \in A^\infty$: $u\ \alpha = u$

The relation of extendability is then defined by :

$$\alpha \leq \beta \iff \exists\, \gamma : \alpha\, \gamma = \beta$$

We say that α is extendable into β iff $\alpha \leq \beta$.

Clearly $\alpha \leq \beta \iff \alpha \in FG(\beta)$ or $\alpha = \beta \in A^\omega$.

Some equalities and equivalences hold

$$FG(FG(\alpha)) = FG(\alpha) \quad (\text{where } FG(FG(\alpha)) = \cup \{FG(\beta) \mid \beta \in FG(\alpha)\}$$

$$\alpha \leq \beta \iff FG(\alpha) \subseteq FG(\beta)$$

$$\alpha = \beta \iff FG(\alpha) = FG(\beta)$$

$$\alpha \in A^{*} \iff \text{card } (FG(\alpha)) < \infty$$

Define an FG-set as any set $L \subseteq A^{\infty}$ which is totally ordered by the relation $\leq$. In other words :

$$L \text{ is an FG-set} \iff \forall\, \alpha,\beta \in L \text{ either } \alpha \leq \beta \text{ or } \beta \leq \alpha.$$

Property 1 : For every FG-set $L \subseteq A^{\infty}$, $L \neq \emptyset$, there exists a unique word $\alpha \in A^{\infty}$ such that :

$$L \subseteq FG(\alpha) \text{ and for all } \beta \quad L \subseteq FG(\beta) \Rightarrow \alpha \leq \beta$$

This word α is denoted Sup (L) and characterized by :

$$FG(\text{Sup}(L)) = FG(L) = \cup \{FG(\alpha) \mid \alpha \in L\}.$$

We assume the above property to be well-known and introduce a few more useful notations :

$$\alpha \in FG(\beta) \Rightarrow \alpha \in A^{*} \text{ and } \exists\,!\, \gamma : \quad \alpha\gamma = \beta$$

This unique element γ is denoted $(\beta:\alpha)$.

If $L \subseteq A^{\infty}$ is an infinitary language and $f \in A^{*}$ a finite word :

$$(L:f) = \{(\beta:f) \mid \beta \in L \text{ and } f \leq \beta\}$$

Clearly we have :

$$f \in L \iff \varepsilon \in (L:f)$$

L is an FG-set implies $(L:f)$ is an FG-set and if $(L:f) \neq \emptyset$

$$f \text{ Sup } (L:f) = \text{Sup}(L)$$

This last assertion comes from the fact that :

$$FG(L:f) = (FG(L):f)$$

In this paragraph we extend the well-known notions just recalled to multiwords and relations.

If $A_1,\ldots,A_k$ are alphabets, it is convenient to denote $\mathcal{A} = A_1^\infty \times \ldots \times A_k^\infty$. Elements in $\mathcal{A}$ are called multiwords (or k-words if we wish to specify the arity) : they are k-uples of the form :

$$\vec{\alpha} = \langle\alpha_1, \ldots, \alpha_k\rangle \quad \text{where } \forall\, i \in [k] \quad \alpha_i \in A_i^\infty$$

We write also $\alpha_i = \pi_i(\vec{\alpha})$.

The multiword $\vec{\alpha}$ is finite iff $\pi_i(\vec{\alpha}) \in A_i^*$ for all $i \in [k]$ and if $\vec{\alpha}$ is finite we define its maximal length

$$|\vec{\alpha}| = \max\,\{\pi_i(\vec{\alpha}) \mid i \in [k]\}$$

The multiword $\vec{\alpha}$ is infinite iff $\pi_i(\vec{\alpha}) \in A_i^\omega$ for at least one i.
Its maximal length $|\vec{\alpha}|$ is then set to be infinite $|\vec{\alpha}| = \infty$.
The minimal length of a multiword is :

$$||\vec{\alpha}|| = \min\,\{\pi_i(\vec{\alpha}) \mid i \in [k]\}$$

This minimal length is finite unless $\pi_i(\vec{\alpha}) \in A_i^\omega$ for all $i \in [k]$
If this is the case $\vec{\alpha}$ is said to be totally infinite.

We denote :

$$\mathcal{A}^{\text{fin}} = \{\vec{\alpha} \in \mathcal{A} \mid |\vec{\alpha}| < \infty\}$$
$$\mathcal{A}^{\text{inf}} = \{\vec{\alpha} \in \mathcal{A} \mid |\vec{\alpha}| = \infty\}$$
$$\mathcal{A}^{\text{tinf}} = \{\vec{\alpha} \in \mathcal{A} \mid ||\vec{\alpha}|| = \infty\}$$

One multiplies multiwords component wise :

$$\vec{\alpha}\,\vec{\beta} = \langle\alpha_1\,\beta_1, \ldots, \alpha_k\,\beta_k\rangle$$

and denote $\vec{\varepsilon}$ the multiword each component of which is ε so that :

$$\vec{\varepsilon}\,\vec{\alpha} = \vec{\alpha}\,\vec{\varepsilon} = \vec{\alpha} \quad \text{for all } \vec{\alpha}.$$

We first define the relation of entendability :

$$\vec{\alpha} \le \vec{\beta} \iff \exists\, \vec{\gamma} \quad \vec{\alpha}\,\vec{\gamma} = \vec{\beta}$$
$$\iff \forall\, i \in [k] \quad \alpha_i \le \beta_i.$$

We denote for all $\vec{\alpha} \in \mathscr{A}$:

$$PP(\vec{\alpha}) = \{\vec{f} \in \mathscr{A}^{fin} \mid \vec{f} \leq \vec{\alpha}\}$$

We clearly have :

$$PP(\vec{\alpha}) = FG(\alpha_1) \times \dots \times FG(\alpha_k)$$

and if $\vec{f} \in PP(\vec{\alpha})$ there exists a unique $\vec{\beta} \in \mathscr{A}$ denoted $(\vec{\alpha}:\vec{f})$ such that $\vec{\alpha}\,\vec{\beta} = \vec{\alpha}(\vec{\alpha}:\vec{f}) = \vec{\alpha}$.

Obviously $\pi_i(\vec{\alpha}:\vec{f}) = (\alpha_i:f_i)$ for all $i \in [k]$.

If we now define a PP-set as an relation $R \subseteq \mathscr{A}$ which is totally ordered by $\leq$ we can state a property which is very similar to property 1.

<u>Property 2</u> : For every non empty PP-relation $R \subseteq \mathscr{A}$ there exists a unique multiword $\vec{\alpha} \in \mathscr{A}$ such that :

$$R \subseteq PP(\vec{\alpha}) \text{ and for all } \vec{\beta} \in \mathscr{A}\ R \subseteq PP(\vec{\beta}) \Rightarrow \vec{\alpha} \leq \vec{\beta}$$

This $\vec{\alpha}$ is denoted Sup(R) and is characterized by :

$$PP(Sup(R)) = PP(R) = \cup \{PP(\vec{\alpha}) \mid \vec{\alpha} \in R\}..$$

<u>Proof</u> : If R is a PP-set then for all $i \in [k]$
$\pi_i(R) = \{\pi_i(\vec{\alpha}) \mid \vec{\alpha} \in R\}$ is an FG-language and $R \neq \emptyset \iff \pi_i(R) \neq \emptyset$ for all i.
$Sup(R) = \langle Sup(\pi_1(R)), \dots, Sup(\pi_k(R))\rangle$ has all the desired properties. □

Many rather obvious identities and equivalences can be stated :

$$PP(PP(\vec{\alpha})) = PP(\vec{\alpha})$$

$$\vec{\alpha} \leq \vec{\beta} \iff PP(\vec{\alpha}) \subseteq PP(\vec{\beta})$$

$$\vec{\alpha} = \vec{\beta} \iff PP(\vec{\alpha}) = PP(\vec{\beta})$$

$$\vec{\alpha} = Sup\ (PP(\vec{\alpha}))\ \ (PP(\vec{\alpha}) \text{ is clearly a non empty PP-set})$$

$$\vec{f}\ Sup\ (R:\vec{f}) = Sup\ (R) \text{ if R is a PP-set and } R:\vec{f} \neq \emptyset$$

A major difficulty when dealing with multiwords and relations is that we have to distinguish between the two relations "$\vec{\alpha}$ is extendable into $\vec{\beta}$" and the relation we now define "$\vec{\alpha}$ is a left factor of $\vec{\beta}$". For $\vec{\alpha} \in \mathscr{A}$ define

$$\vec{\alpha}[p] = \langle\alpha_1[p], \dots, \alpha_k[p]\rangle$$

$$FG(\vec{\alpha}) = \{\vec{\alpha}[p] \mid p \in \mathbb{N}\}$$

It is no longer true that :

$$\vec{\alpha} \le \vec{\beta} \iff \vec{\alpha} \in FG(\vec{\beta}) \text{ or } \vec{\alpha} = \vec{\beta} \in \mathcal{R}^{\inf}$$

(one can provide immediate exemples).

We have the following inclusions identities and implications :

$$FG(FG(\vec{\alpha})) = FG(\vec{\alpha}) \tag{1}$$

Proof : For all $\vec{\alpha} \in \mathcal{R}$, n, p $\in$ IN

$$(\vec{\alpha}[n])\ [p] = \vec{\alpha}[\min(n,p)] \qquad \square$$

$$FG(\vec{\alpha}) = FG(\vec{\beta}) \iff \vec{\alpha} = \vec{\beta} \tag{2}$$

Proof : For all i $\in$ [k] we have :

$$\pi_i(FG(\vec{\alpha})) = FG(\pi_i(\vec{\alpha}))$$

for if $f_i \in FG(\alpha_i)$ and $n = |f_i|$ then $\pi_i(\vec{\alpha}[n]) = \alpha_i[n] = f$ which proves one inclusion. The reverse inclusion is obvious.

Then $FG(\vec{\alpha}) = FG(\vec{\beta})$ implies for all i $\in$ [k]

$$\pi_i(FG(\vec{\alpha})) = \pi_i(FG(\vec{\beta}))$$

which implies $FG(\pi_i(\vec{\alpha})) = FG(\pi_i(\vec{\beta}))$.

Thus for all i $\in$ [k] $\pi_i(\vec{\alpha}) = \pi_i(\vec{\beta})$ and $\vec{\alpha} = \vec{\beta}$ $\square$

$$FG(\vec{\alpha}) \subseteq FG(\vec{\beta}) \Rightarrow \vec{\alpha} \le \vec{\beta} \tag{3}$$

The proof is the same as above.

The interesting fact is that the reverse is not true and we can only state if we define for all R, R' $\subseteq \mathcal{R}$

$$R \le R' \iff \forall\ \vec{\alpha} \in R\ \exists\ \vec{\beta} \in R' \quad \vec{\alpha} \le \beta$$

$$\vec{\alpha} \le \vec{\beta} \iff FG(\vec{\alpha}) \le FG(\vec{\beta}) \tag{4}$$

Proof : $\alpha_i \le \beta_i \Rightarrow \alpha_i[n] \le \beta_i[n]$ for all n.

Whence $\vec{\alpha} \le \vec{\beta} > \vec{\alpha}[n] \le \vec{\beta}[n]$ for all n.

Conversely $FG(\vec{\alpha}) \le FG(\vec{\beta})$ implies.

$\pi_i(FG(\vec{\alpha})) \le \pi_i(FG(\vec{\beta}))$ for all i since $f_i \in \pi_i(FG(\vec{\alpha})) \Rightarrow f_i = \pi_i(\vec{f}\ )$ for some $\vec{f} \in FG(\vec{\alpha})$.

There exists then $\vec{g} \in FG(\vec{\beta})$ such that $\vec{f} \leq \vec{g}$ and clearly $f_i = \pi_i(\vec{f}) \leq \pi_i(\vec{g}) \in \pi_i(FG(\vec{\beta}))$.
For all i we then have $FG(\pi_i(\vec{\alpha})) \leq FG(\pi_i(\vec{\beta}))$ which obviously implies $FG(\pi_i(\vec{\alpha})) \subseteq FG(\pi_i(\vec{\beta}))$ since $f \leq g \in FG(\pi_i(\vec{\beta})) \Rightarrow f \in FG(g) \subseteq FG(\pi_i(\vec{\beta}))$.
Eventually for all i $\pi_i(\vec{\alpha}) \leq \pi_i(\vec{\beta})$ and $\vec{\alpha} \leq \vec{\beta}$. □

For all $\vec{\alpha} \in \mathscr{R}$ $FG(\vec{\alpha})$ is a non empty PP set
and we have $\vec{\alpha} = \mathrm{Sup}\ (FG(\vec{\alpha}))$. (5)

An immediate consequence is that $PP(\vec{\alpha}) = PP(FG(\vec{\alpha})) = FG(PP(\vec{\alpha}))$.

Proof. Clearly for all n, p

$$\vec{\alpha}[\min(n,p)] \leq \vec{\alpha}[\max(n,p)]$$

Suppose $\vec{\beta} = \mathrm{Sup}(FG(\vec{\alpha}))$. Then for all i :

$$\pi_i(\vec{\beta}) = \mathrm{Sup}\ (\pi_i(FG(\vec{\alpha}))) = \mathrm{Sup}\ (FG(\pi_i(\vec{\alpha}))) = \pi_i(\vec{\alpha}). \qquad \square$$

III - CONVERGING SEQUENCES, ADHERENCES AND CENTERS

In A^∞ there exists a very natural metric topology defined by the distance $d : A^\infty \times A^\infty \to \mathbb{R}_+$ given by :

$$d(\alpha,\beta) = 2^{-\min\{n \mid \alpha[n] \neq \beta[n]\}} \text{ if } \alpha \neq \beta$$

$$d(\alpha,\beta) = 0 \text{ if } \alpha = \beta$$

A^∞ equiped with the metric d and the topology induced by d is a complete metric space which is also compact.

It may be useful to recall a few properties of this d-topology. The sequence α_n, $n \in \mathbb{N}_+$ converges iff it is d-Cauchy, i.e. satisfies :

$$\forall\ n \in \mathbb{N}\ \exists\ N \in \mathbb{N}\quad p,q \geq N \Rightarrow \alpha_p[n] = \alpha_q[n]$$

Its limit $\alpha_o = \lim \alpha$ is then the unique word in A^∞ such that :

$$\forall\ n \in \mathbb{N}\ \exists\ N \in \mathbb{N}\quad p \geq N \Rightarrow \alpha_p[n] = \alpha_o[n]$$

And we can see that $\lim \alpha$ is a finite word $\alpha_o \in A^*$ only if the sequence α_n is stationary :

$$\forall\ n \in \mathbb{N}\ \exists\ N \in \mathbb{N}\quad p \geq N \Rightarrow \alpha_p = \alpha_o$$

A property which will play a role in the sequel is that :

Property 3 : From every sequence α_n, $n \in \mathbb{N}_+$ of words in A^∞ one can extract a converging subsequence.

Proof : The statement means that one can find an infinite subset N' of $\mathbb{N}$ such that α_n, $n \in N'$ converges, i.e. satisfies the d-Cauchy condition :

$$\forall\ n\ \exists\ N \quad p,q \geq N \text{ and } p,q \in N' \Rightarrow \alpha_p[n] = \alpha_q[n]$$

Let us remark immediately that the convergence of α_n, $n \in \mathbb{N}_+$ implies the convergence of α_n, $n \in N'$ for all infinite subset N' of $\mathbb{N}_+$.

Two cases arise :

- The sequence of lengths $|\alpha_n|$, $n \in \mathbb{N}_+$ is ultimately bounded i.e. satisfies :

$$\exists\, N \in \mathbb{N}_+,\ M \in \mathbb{N}\ \forall n \geq N \quad |\alpha|_n \leq M$$

Then for all $n \geq N$, α_n belongs to the finite set :

$$A^{\leq M} = \{f \in A^* \mid |f| \leq M\}$$

And certainly there exists $f_o \in A \leq M$ such that :

$$N' = \{n \geq N \mid \alpha_n = f_o\} \text{ is infinite.}$$

Then α_n, $n \in N'$ converges towards f_o.

- The sequence $|\alpha_n|$ is not ultimately bounded i.e.

$$\forall\, N \in \mathbb{N}_+,\ M \in \mathbb{N} \quad \exists\, n \geq N \quad |\alpha_n| > M$$

Define for all $m \in \mathbb{N}_+$:

$$E_m = \{f \in A^m \mid \text{card } \{n \mid f \leq \alpha_n\} = \infty\}$$

This set is for all m finite and non empty : finite for $E_m \subseteq A^m$ which is finite and non empty for there are infinitely many n's for which $|\alpha_n| \geq m$ and thus has a left factor in A^m. Whence certainly at least one $f \in A^m$ is a left factor of α_n for infinitely many n's.

Obviously if $f \in E_{m+1}$, $f[m] \in E_m$.

By Koenig's lemma one can find a sequence f_m, $m \in \mathbb{N}_+$ such that $f_m \in E_m$ and $f_m = f_{m+1}[m]$ for all m.

This sequence obviously converges towards $u \in A^\omega$ such that $u[m] = f_m$ for all m.

Now we can build an infinite strictly increasing sequence of integers $n_1, n_2, \ldots, n_m, \ldots$ such that for all m $f_m \leq \alpha_{n_m}$ and $n_m < n_{m+1}$. The sequence α_{n_m} converges towards u. □

It will be quite convenient to use increasing sequences of words. In fact any increasing sequence α_n, $n \in \mathbb{N}_+$ such that :

$$\forall n \quad \alpha_n \leq \alpha_{n+1} \text{ is converging to Sup } \{\alpha_n\}.$$

Closed subsets : an infinitary language $L \subseteq A^{\infty}$ is closed iff it contains the limits of sequences α_n, $n \in \mathbb{N}_+$ where $\alpha_n \in L$. From the definition of $\alpha_o = \lim \alpha_n$ it is clear that $\alpha_o \in A^* \Leftrightarrow \alpha_o \in L^{fin} = L \cap A^*$. And $\alpha_o \in A^{\omega} \Leftrightarrow \forall n\, \exists p\ \alpha_o[n] = \alpha_q[n]$ for all $q > p$. It follows that all the limits of sequences of words in L are finite or belong to $Adh(L) = \{u \in A^{\omega} \mid FG(u) \subseteq FG(L)\}$. And $u \in Adh(L) \Rightarrow \forall n\, \exists\, \alpha_n \in L$ $u[n] = \alpha_n[n] \Rightarrow u = \lim \alpha_n$.

We can state the :

Property 4 : The infinitary language $L \subseteq A^{\infty}$ is closed iff :

$$Adh(L) \subseteq L$$

The topological closure of L i.e. the smallest closed language containing L is $\bar{L} = L \cup Adh(L)$.

The set Adh(L) is called the adherence of L in [3]. Many properties of this adherence have been established in the same papers : a major tool to study the adherence is the center of a language defined as the set of left factors of the adherence :

$$\underline{L} = FG(Adh(L))$$

The center is characterized by :

$$\underline{L} = FG(L^{inf}) \cup \{f \in FG(L^{fin}) \mid (L:f) \text{ is infinite}\}$$

In this writing $L^{inf} = L \cap A^{\omega}$, $L^{fin} = L \cap A^*$.

And we have :

$$Adh(L) = Adh(\underline{L})$$

$$Adh(L_1) = Adh(L_2) \Leftrightarrow \underline{L_1} = \underline{L_2}$$

IV - ADHERENCES AND CENTERS OF RELATIONS

The cartesian product $\mathcal{A} = A_1^\infty \times \ldots \times A_k^\infty$ can be equiped with the distance d : $\mathcal{A}\times\mathcal{A}\to \mathbb{R}_+$ given by :

$$d(\vec{\alpha},\vec{\beta}) = 2^{-\min\{n \mid \vec{\alpha}[n] \neq \vec{\beta}[n]\}} \text{ if } \vec{\alpha} \neq \vec{\beta}$$

$$= 0 \text{ if } \vec{\alpha} = \vec{\beta}$$

Clearly :

$$d(\vec{\alpha},\vec{\beta}) = \max\{d(\alpha_i,\beta_i) \mid i \in [k]\}$$

Proof : If $\vec{\alpha} \neq \vec{\beta}$ and n is the smallest integer such that $\vec{\alpha}[n] \neq \beta[n]$ we have :

$$\vec{\alpha}[n-1] = \vec{\beta}[n-1] \text{ and for at least one } i$$

$$\pi_i\,(\vec{\alpha}[n]) \neq \pi_i\,(\vec{\beta}[n]).$$

Thus for all i $d(\alpha_i,\beta_i) \leq 2^{-n}$ and for a least one i $d(\alpha_i,\beta_i) = 2^{-n}$. □

The topology induced by d on $\mathcal{A}$ is the product topology.

Thus the sequence $\vec{\alpha}_n$, $n \in \mathbb{N}$ of multiwords converges iff and only if it is d-Cauchy i.e. :

$$\forall n\ \exists N \quad p,q > N \Rightarrow \vec{\alpha}_p[n] = \vec{\alpha}_q[n]$$

and this happens iff for all i :

$$\pi_i(\vec{\alpha}_n),\ n \in \mathbb{N} \text{ is d-Cauchy}$$

If $\vec{\alpha}_n$, $n \in \mathbb{N}$ is d-Cauchy it converges towards $\vec{\alpha}_o$, which is unique, such that :

$$\forall n\ \exists N \quad p > N \Rightarrow \vec{\alpha}_o[n] = \vec{\alpha}_p[n]$$

and if $\vec{\alpha}_o = \lim \vec{\alpha}_n$ we have :

$$\pi_i\,(\lim \vec{\alpha}_n) = \lim\,(\pi_i(\vec{\alpha}_n)) \text{ for all } i.$$

More precisely $\vec{\alpha}_n$ converges iff $\pi_i(\vec{\alpha}_n)$ converges for all i and $\lim\,(\vec{\alpha}_n) = \langle \lim \pi_1(\vec{\alpha}_n), \ldots, \lim \pi_k(\vec{\alpha}_n)\rangle$.

We can entend property 3 and state.

Property 5 : From any sequence $\vec{\alpha}_n$, $n \in \mathbb{N}$ of multiwords. We can extract a converging subsequence.

Proof : We extract the converging subsequence :

$$\pi_1\ (\vec{\alpha}_n),\ n \in N_1 \text{ from } \pi_1\ (\vec{\alpha}_n)$$

Then we can extract $\pi_2\ (\vec{\alpha}_n)$, $n \in N_2$ from the sequence $\pi_2\ (\vec{\alpha}_n)$, $n \in N_1$: this means that we can find an infinite subset N_2 of N_1 such that :

$$\pi_2\ (\vec{\alpha}_n),\ n \in N_2 \text{ converges.}$$

By induction we thus build $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_k$ such that $\pi_i(\vec{\alpha}_n)$, $n \in N_i$ converges for all i.

By a remark made above $N_k \subseteq N_i$ and N_k infinite implies that $\pi_i(\vec{\alpha}_n)$, $n \in N_k$ converges.

Thus $\vec{\alpha}_n$, $n \in N_k$ converges. □

Closed relations

Let $R \subseteq \mathcal{A}$ be an infinitary relation.

It is closed iff it contains all the limits of converging sequences of multiwords in R.

As in the case of languages $\vec{\alpha}_n$, $n \in \mathbb{N}$ converges towards $\vec{\alpha}_o \in \mathcal{A}^{fin}$ iff $\vec{\alpha}_n = \vec{\alpha}_o$ for all sufficiently large n's and this implies $\vec{\alpha}_o \in R^{fin} = R \cap \mathcal{A}^{fin}$.

And the set of infinite limits :

$$\{\vec{\alpha}_o \in \mathcal{A}^{inf} \mid \vec{\alpha}_o = \lim \vec{\alpha}_n,\ \alpha_n \in R\}$$

is equal to :

$$Adh(R) = \{\vec{\alpha} \in \mathcal{A}^{inf} \mid FG(\vec{\alpha}) \subseteq FG(R)\}$$

We call Adh(R) the adherence of R.

Property 6 : The infinitary relation R is closed (in the d-topology) iff $Adh(R) \subseteq R$. The topological closure of R is $\bar{R} = R \cup Adh\ (R)$.

As in the case of languages a major tool to study adherences and closed relations is the notion of center.

The center of a relation $R \subseteq \mathcal{R}$ is $\underline{R} = FG(Adh(R))$. We first characterize the center of a finitary relation.

<u>Property 7</u> : If $R \subseteq \mathcal{R}^{fin}$ is finitary

$$\underline{R} = \{\vec{f} \in \mathcal{R}^{fin} \mid \text{card } \{\vec{g} \in R \mid \vec{f} \in FG(\vec{g})\} = \infty\}$$

Proof : We denote $FG^{-1}(\vec{f})$ the set of all $\vec{g}$ such that

$$\vec{f} \in FG(\vec{g}).$$

Suppose first $\vec{f} \in FG(\vec{\alpha})$ for some $\vec{\alpha} \in Adh(R)$. For all n $\vec{\alpha}[n] \in FG(R)$ i.e. for all n there exists $\vec{g}_n \in R$ such that $\vec{\alpha}[n] = \vec{g}_n[n]$.

The fact that $\vec{\alpha} \in \mathcal{R}^{inf}$ implies that :

$|\vec{\alpha}[n]| = n$ whence $|\vec{g}_n| \geq n$ and the set of $\vec{g}_n$'s is infinite.

If $|\vec{f}| = p$, $\vec{f} = \vec{\alpha}[p] = (\vec{\alpha}[n])\ [p]$ for all $n \geq p$.

Thus $\vec{f} = (\vec{g}_n[n])\ [p] = \vec{g}_n[p]$ for all $n \geq p$. And $FG^{-1}(\vec{f}) \cap R$ is infinite.

Conversely suppose $FG^{-1}(\vec{f}) \cap R$ is infinite : this implies $FG^{-1}(\vec{f}) \cap FG(R)$ is infinite since $R \subseteq \mathcal{R}^{fin} \Rightarrow R \subseteq FG(R)$.

Consider then the following sequence of sets indexed by $m \geq n = |\vec{f}|$:

$$E_m = \{\vec{g} \in FG(R) \mid \vec{f} \in FG(\vec{g}) \text{ and } |\vec{g}| = m\}$$

This set is for all m finite (obviously) and non empty for $FG^{-1}(\vec{f}) \cap FG(R)$ implies the existence of $\vec{g}$ in this set with arbitrarily large $|\vec{g}|$. And if $\vec{f} = \vec{g}[n]$, $|\vec{g}| \geq m$ then $\vec{f} = (\vec{g}[m])\ [n]$ whence $\vec{g}[m] \in E_m$ since $|\vec{g}[m]| = m$.

Now if $\vec{g} \in E_{m+1}$ clearly $\vec{g}[m] \in E_m$.

We can find, by Koenig's lemma, $\vec{g}_m \in E_m$ such that for all m, $\vec{g}_m = \vec{g}_{m+1}[m]$.

The increasing sequence $\vec{g}_m$ converges towards some $\vec{\alpha} \in Adh(R)$ since $\vec{\alpha}[m] = \vec{g}_m \in FG(R)$ for all m.

And we have $\vec{f} = \vec{\alpha}[n] \in FG(Adh(R))$. □

One could have been tempted to write :

$$\underline{R} = \{\vec{f} \in FG(R) \mid (R:\vec{f}) \text{ is infinite}\}.$$

This is false as proved by the exemple :

$$R = \{<a^n, b^{2n}> \mid n \in \mathbb{N}\}$$

$$FG(R) = \{<a^n, b^p> \mid n \le p \le 2n\}$$

Clearly $R : <a^n, b^p>$ is infinite for all n, $n \le p \le 2n$ for :

$$<a^n, b^p> <a^q, b^{2n+2q-p}> = <a^{n+q}, b^{2n+2q}> \in R$$

The adherence of R is $<a^\omega, b^\omega>$ and the center is

$$\underline{R} = <a^n, b^n>.$$

One can check that $FG^{-1}(<a^n, b^p>) \cap R$ is infinite iff $n = p$: indeed $<a^n, b^n> \in FG(<a^m, b^{2m}>)$ for all $m \ge n$ and $<a^n, b^p>$, $n < p \le 2n$ is a left factor of $<a^n, b^m>$, $p \le m \le 2n$. □

Now the center of an infinitary relation is :

$$\underline{R} = FG(R^{inf}) \cup \underline{R}^{fin} \text{ where :}$$

$$R^{inf} = R \cap \mathcal{R}^{inf} \text{ and } R^{fin} = R \cap \mathcal{R}^{fin}.$$

The properties of the center we stated for languages are preserved :

$$Adh(R) = Adh(\underline{R})$$

$$Adh(R_1) = Adh(R_2) \iff \underline{R}_1 = \underline{R}_2.$$

We also have for every finitary relation R

$$Adh(R) \neq \emptyset \iff \underline{R} \neq \emptyset \iff R \text{ is infinite.}$$

Other obvious properties are :

$$Adh(R) = Adh(FG(R))$$

$$\underline{R} = \underline{FG(R)} = FG(\underline{R})$$

And the following exemple proves that Adh (PP(R)) is usually strictly greater than Adh(R) :

$$R = \{<a^n, b^n> \mid n \in \mathbb{N}\}$$

$$\mathrm{Adh}(R) = \{\langle a^\omega, b^\omega\rangle\}$$

$$PP(R) = a^* \times b^*$$

$$\mathrm{Adh}(PP(R)) = a^* \times \{b^\omega\} \cup \{a^\omega\} \times b^* \cup \{\langle a^\omega, b^\omega\rangle\}$$

Property 8 : $\mathrm{Adh}(PP(R)) = \widehat{PP}(\mathrm{Adh}(R)) \cap \mathcal{R}^{inf}$.

Where $\widehat{PP}\ (\beta) = \{\hat\alpha \mid \hat\alpha \le \hat\beta\}$

Proof : One inclusion is obvious.

To prove the reverse inclusion we remark that :

$\vec f \in \widehat{PP}(R)$ and $|\vec f| = n$ imply that there exists

$\vec g \in FG(R)$ such that $|\vec g| = n$ and $\vec f \le \vec g$.

Indeed : $\vec f \in PP(R) \Rightarrow \exists\ \vec\alpha \in R : \vec f \le \vec\alpha$.

But then : $\vec f[n] = \vec f \le \vec\alpha[n] \in FG(R)$.

Suppose : $\vec\alpha \in \mathrm{Adh}(PP(R))$.

Consider then the sequence of sets :

$$E_n = \{\vec g \in FG(R) \mid \vec\alpha[n] \le \vec g \text{ and } |\vec g| = n\}$$

These sets being finite and non empty and satisfying :

$$\forall n \quad \vec g \in E_{n+1} \Rightarrow \vec g[n] \in E_n$$

we can find, by Koenig's lemma, an infinite sequence g_n with for all n $g_n \in E_n$ and $g_n = g_{n+1}[n]$. This sequence converges to β such that :

$$g_n = \beta[n] \text{ for all } n$$

This limit β belongs to Adh(R) and from $\vec\alpha[n] \le \vec g_n \le \vec\beta$ for all n we deduce :

$$\vec\alpha = \lim \vec\alpha[n] \le \beta \text{ whence } \vec\alpha \in \widehat{PP}(\mathrm{Adh}(R)). \quad \square$$

We can make precise a remark made above.

Property 9 : PP(R) = FG(Adh(PP(R))) is precisely the set of all $\vec f$ such that $R:\vec f$ is infinite.

Proof : One inclusion is obvious. The second one is one more straight foward application of Koenig's lemma. $\square$

V – OPERATIONS ON RELATIONS AND CLOSEDNESS

V.1. Union and product

In [ ] we have studied the topological closure of the union and product of two languages and the star of a language.

There are many more operations on relations and in this paragraph, we shall establish a number of formulae to compute the topological closure of the result of these operations in terms of the topological closure of the relations on which they are performed.

The simplest cas is the union which gives rise to :

Property 10 : $\overline{R_1 \cup R_2} = \overline{R_1} \cup \overline{R_2}$.

Proof : The union of two closed relations is closed.
Thus $R_1 \subseteq \overline{R_1}$ and $R_2 \subseteq \overline{R_2}$ imply :

$$R_1 \cup R_2 \subseteq \overline{R_1} \cup \overline{R_2} \quad \text{and} \quad \overline{R_1 \cup R_2} \subseteq \overline{R_1} \cup \overline{R_2}$$

Conversely $R_1 \subseteq R_1 \cup R_2$ and $R_2 \subseteq R_1 \cup R_2$ imply :

$$\overline{R_1} \subseteq \overline{R_1 \cup R_2} \quad \text{and} \quad \overline{R_2} \subseteq \overline{R_1 \cup R_2}$$

for the closure is obviously increasing $R \subseteq R' \Rightarrow \bar{R} \subseteq \bar{R}'$ □

As concerns the product componentwise i.e.

$$R_1 R_2 = \{\vec{\alpha} \vec{\beta} \mid \vec{\alpha} \in R_1 \quad \vec{\beta} \in R_2\}$$

we have the same formula as in the case of languages.

Property 11 : $\overline{R_1 R_2} = \overline{R_1}\, \overline{R_2}$

Proof : In the case of languages $L_1, L_2 \subseteq A^*$, we had proved :

$$\mathrm{Adh}(L_1 L_2) = \mathrm{Adh}(L_1) \cup L_1^{fin} \mathrm{Adh}(L_2)$$

whence :

$$\overline{L_1 L_2} = L_1 L_2 \cup Adh(L_1 L_2)$$

$$= (L_1 \cup Adh(L_1)) (L_2 \cup Adh(L_2)) = \overline{L_1}\, \overline{L_2}$$

since :

$$(L_1 \cup Adh(L_1)) (L_2 \cup Adh(L_2)) = L_1 L_2 \cup Adh\ L_1 \cup L_1^{fin} Adh(L_2)$$

for :

$$L_1^{inf} Adh(L_2) = L_1^{inf} L_2 = L_1^{inf} \subseteq L_1 L_2$$

We cannot really use the same method for relations as for languages since the formula $FG(L_1 \cdot L_2) = FG(L_1) \cup L_1^{fin} FG(L_2)$ is not valid for relations.

We give here a proof which uses as few theorems from topology as possible. It relies on the :

Lemma 1 : If α_n, $n \in \mathbb{N}_+$ and β_n, $n \in \mathbb{N}_+$ are two sequences of words in A^∞ such that the sequence :

$$\gamma_n = \alpha_n \beta_n \text{ converges towards } \gamma_o.$$

There exists an infinite subsets N' of $\mathbb{N}$ such that α_n, $n \in N'$ and β_n, $n \in N'$ are both convergent and their limits α_o, β_o satisfy $\alpha_o \beta_o = \gamma_o$.

Proof : Suppose $\gamma_o = \lim \gamma_n$ is finite $\gamma_o \in A^*$.
We know that γ_n is stationary i.e.

$$\exists N \quad n \geq N \Rightarrow \gamma_n = \gamma_o$$

For all $n \geq N$ we can thus write :

$$\langle\alpha_n, \beta_n\rangle \in \{\langle f_1, f_2\rangle \mid f_1 f_2 = \gamma_o\}$$

Since the set of binary factorisations of γ_o is finite, there exist $\langle f_1, f_2\rangle$ such that $f_1 f_2 = \gamma_o$ and $\langle\alpha_n, \beta_n\rangle = \langle f_1, f_2\rangle$ for infinitely many n's say all $n \in N'$ where N' is an infinite subset of $\mathbb{N}_+$. Clearly then α_n, $n \in N'$ converges towards f_1 and β_n, $n \in N'$ converges towards f_2.

Suppose now γ_o is infinite : two case arise :

1) $|\alpha_n|$ is ultimately-bounded i.e. $|\alpha_n| \le M$ for all sufficiently large $n \in \mathbb{N}$. We have $\forall p\ \exists n_p\ \ n \ge n_p \Rightarrow \gamma_n[p] = \gamma_o[p]$ and since $\gamma_n = \alpha_n\ \beta_n (\alpha_n\ \beta_n)[p] = \gamma_o[p]$. But $|\alpha_n| \le M$ whence for all $p \ge M$ and $n \ge n_p$:

$$\gamma_n[p] = \alpha_n \cdot \beta'_n \quad \text{with} \quad \beta'_n \le \beta_n$$

For infinitely many n's $\alpha_n = f \in A \le M$, write :

$$\alpha_n = f \text{ and } \gamma_o[p] = f\ \beta_n[p-|f|]$$

Clearly if N' is this infinite set of n's

$$\alpha_n,\ n \in N' \text{ converges towards } f \text{ and}$$

$$\beta_n,\ n \in N' \text{ converges towards } \gamma_o : f$$

2) The sequence $|\alpha_n|$ is not ultimately bounded. We can extract α_n, $n \in N'$ which converges towards $u \in A^\omega$.

From $\alpha_n \le \gamma_n$ for all n we can conclude $\lim \alpha_n \le \lim \gamma_n$ whence $u \le \gamma_o$ which implies $u = \gamma_o$. We then entract from β_n, $n \in N'$ any converging sequence β_n, $n \in N''$ and clearly both α_n, $n \in N''$ and β_n, $n \in N''$ converge with $\lim \alpha_n \lim \beta_n = u \lim \beta_n = u = \gamma_o$. □

By a simple induction we extend this lemma to sequences of multiwords and this proves the inclusion :

$$\overline{L_1\ L_2} \subseteq \overline{L_1}\ \overline{L_2}$$

The reverse inclusion comes from the simple fact :

<u>Lemma 2</u> : If α_n, $n \in \mathbb{N}_+$ and β_n, $n \in \mathbb{N}_+$ converge towards α_o and β_o then $\alpha_n\ \beta_n$, $n \in \mathbb{N}$ converges towards $\alpha_o\ \beta_o$. The same holds for sequences of multiwords.

The proof is straight forward. □

We can now give a formula to compute $\mathrm{Adh}(R_1\ R_2)$:

$$\mathrm{Adh}(R_1\ R_2) = \mathrm{Adh}(R_1)\ R_2 \cup R_1\ \mathrm{Adh}(R_2) \cup \mathrm{Adh}(R_1)\ \mathrm{Adh}(R_2)$$

This comes from :

$$\begin{aligned}\mathrm{Adh}(R_1\ R_2) &= \overline{R_1\ R_2}^{\,\inf} \\ &= ((R_1 \cup \mathrm{Adh}(R_1))\ (R_2 \cup \mathrm{Adh}(R_2)))^{\inf}\end{aligned}$$

5.2. Projections and slices of relations

Let $\vec{\alpha} \in \mathcal{A} = A_1^\infty \times \ldots \times A_k^\infty$ and I be a subset of $[k] = \{1,2,\ldots,k\}$ which can always be written $I = \{i_1,\ldots,i_\ell\}$ with $i_1 < i_2 < \ldots < i_\ell$.

Then $\pi_I(\vec{\alpha})$ is the multiword in $A_{i_1}^\infty \times \ldots \times A_{i_\ell}^\infty$ given by :

$$\pi_I(\vec{\alpha}) = <\pi_{i_1}(\vec{\alpha}),\ldots, \pi_{i_\ell}(\vec{\alpha})>$$

π_I is a projection of $\mathcal{A}$ onto $\mathcal{A}_I = A_{i_1}^\infty \times \ldots \times A_{i_j}^\infty$. We shall need an operation of recomposition.

If I,J are disjoint subsets of [k], we denote I + J their union and if $\vec{\alpha} \in \mathcal{A}_I$, $\vec{\beta} \in \mathcal{A}_J$ we denote $\vec{\alpha} \times \vec{\beta}$ the multiword in $\mathcal{A}_{I+J}$ given by :

$$\pi_i(\vec{\alpha}\times\vec{\beta}) = \pi_i(\vec{\alpha}) \text{ if } i \in I$$

$$= \pi_i(\vec{\beta}) \text{ if } i \in J$$

These operations are entended to relations :

$$\pi_I(R) = \{\pi_i(\vec{\alpha}) \mid \vec{\alpha} \in R\}$$

$$R_1 \times R_2 = \{\vec{\alpha} \times \vec{\beta} \mid \vec{\alpha} \in R_1 , \vec{\beta} \in R_2\} \text{ if}$$

$R_1 \subseteq \mathcal{A}_I$, $R_2 \subseteq \mathcal{A}_J$ where I and J are disjoint subsets of [k].

We now define slices of a relation :

Let $\vec{\alpha}$ be an element of $\mathcal{A}_I$ for some non empty subset I of [k]. We denote $R(I,\vec{\alpha})$, and call slice of R along $\vec{\alpha}$, the relation in $\mathcal{A}_{[k]\setminus I}$ given by :

$$\vec{\beta} \in R(I,\vec{\alpha}) \iff \vec{\beta} \times \vec{\alpha} = \vec{\alpha} \times \vec{\beta} \in R.$$

We have some obvious identities.

For all non empty subset I of [k]

$$R = \cup \{R(I,\vec{\alpha}) \times \{\vec{\alpha}\} \mid \vec{\alpha} \in \mathcal{A}_I\}$$

and since obviously :

$$R(I,\vec{\alpha}) \neq \emptyset \iff \vec{\alpha} \in \pi_I(R)$$

we can rewrite this identity :

$$R = \cup \{R(I,\vec{\alpha}) \times \{\vec{\alpha}\} \mid \vec{\alpha} \in \pi_I(R)\}.$$

A relation is said to be decomposable iff there exist a partition of $[k]$ in $[k] = I_1 + \ldots + I_\ell$, where $I_1, \ldots, I_\ell$ are non empty pairwise disjoint subsets such that :

$$R = \pi_{I_1}(R) \times \pi_{I_2}(R) \times \ldots \times \pi_{I_\ell}(R)$$

A cartesian relation is a relation R which is equal to the product of its components $R = \pi_1(R) \times \ldots \times \pi_k(R)$.

We establish a few easy properties which will be useful later on :

<u>Property 12</u> : $\overline{\pi_I(R)} = \pi_I(\overline{R})$

<u>Proof</u> : Suppose $\vec{\alpha}_n$, $n \in \mathbb{N}_+$, $\alpha_n \in R$ converges towards $\vec{\alpha}_o$ i.e. $\forall p \; \exists N \quad n \geq N \Rightarrow \alpha_o[p] = \alpha_n[p]$.

Then $\pi_I(\vec{\alpha}_n)$ converges towards $\pi_I(\vec{\alpha}_o)$ since $\pi_I(\vec{\alpha}[p]) = (\pi_I(\vec{\alpha}))\,[p]$ for all p.

Conversely if $\vec{\beta}_n$, $n \in \mathbb{N}_+$, $\vec{\beta}_n \in \pi_I(R)$ converges towards $\vec{\beta}_o$ we consider for all n some $\vec{\alpha}_n \in R$ such that $\vec{\beta}_n = \pi_I(\vec{\alpha}_n)$.

From the sequence of multiwords $\vec{\alpha}_n$, $n \in \mathbb{N}_+$ we can entract a converging subsequence $\vec{\alpha}_n$, $n \in N'$ for some infinite subset N' of $\mathbb{N}_+$.

Suppose $\vec{\alpha}_n$, $n \in N'$ converges towards $\vec{\alpha}_o$: it is easy to see that $\pi_I(\vec{\alpha}_o) = \vec{\beta}_o$ since $\pi_I(\vec{\alpha}_o) = \lim \pi_I(\vec{\alpha}_n) = \lim \vec{\beta}_n$. □

<u>Property 13</u> : $\overline{R_1 \times R_2} = \overline{R_1} \times \overline{R_2}$

<u>Proof</u> : It is immediate if one remarks that for all $\vec{\alpha} \in \mathcal{R}_{I+J}$ and $p \in \mathbb{N}$:

$$\vec{\alpha}[p] = (\pi_I(\vec{\alpha}))\,[p] \times (\pi_J(\vec{\alpha}))\,[p]\,. \;\square$$

In order to characterize the closure of slices we have to introduce the notion of a limit of a sequence of sets or relations.

If R_n, $n \in \mathbb{N}_+$ is a sequence of relations, its limit is defined by :

$$\lim R_n = \{\lim \vec{\beta}_n \mid \vec{\beta}_n \text{ converges and } \forall n \ \vec{\beta}_n \in R_n\}$$

Then we can state :

Property 14 : $\bar{R}(I,\vec{\alpha}_o) = \lim R(I,\vec{\alpha}_n)$ for all sequence $\vec{\alpha}_n$ converging towards $\vec{\alpha}_o$ and satisfying $\vec{\alpha}_n \in \pi_I(R)$ for all n.

Proof : Let $\vec{\beta}_o \in \bar{R}(I,\alpha_o)$ i.e. $\vec{\alpha}_o \times \vec{\beta}_o$ is the limit of some sequence $\vec{\gamma}_n$, $\vec{\gamma}_n \in R$.

We can write $\vec{\gamma}_n = \pi_I(\vec{\gamma}_n) \times \pi_J(\vec{\gamma}_n)$ if I+J = [k]. Clearly $\pi_J(\vec{\gamma}_n)$ converges towards $\vec{\beta}_o$ and for all n :

$$\pi_J(\vec{\gamma}_n) \in R(I, \pi_I(\vec{\gamma}_n)).$$

The reverse inclusion is also immediate :

Suppose $\vec{\beta}_n \in R(I,\vec{\alpha}_n)$ for all n and $\vec{\alpha}_n \to \vec{\alpha}_o$. Suppose $\vec{\beta}_n$ converges towards $\vec{\beta}_o$. Then $\vec{\alpha}_n \times \vec{\beta}_n$ converges towards $\vec{\alpha}_o \times \vec{\beta}_o \in \bar{R}$ and thus $\vec{\beta}_o \in \bar{R}(I,\vec{\alpha}_o)$. □

An immediate corollary is :

Corollary 1 : If $\alpha_o \in \mathscr{R}^{fin}$ $\bar{R}(I,\vec{\alpha}_o) = \overline{R(I,\vec{\alpha}_o)}$

Proof : $\vec{\alpha}_n \to \vec{\alpha}_o$ implies that $\vec{\alpha}_n$ is stationary whence $R(I,\vec{\alpha}_n)$ is stationary, equal to $R(I,\vec{\alpha}_o)$ for all sufficiently large n's. □

5.3. Infinite stars of relations

The star of the relation $R \subseteq \mathscr{R}$ is defined by

$$R^* = \cup \{R^n \mid n \in \mathbb{N}\}$$

where $R^o = \{\vec{\varepsilon}\}$ by definition and for all n :

$$R^{n+1} = R\, R^n$$

R^ω is the set of all infinite products of the form

$$\vec{\alpha} = \vec{\alpha}_1\, \vec{\alpha}_2 \ldots \vec{\alpha}_n \ldots \text{ where } \vec{\alpha}_i \in R\backslash\{\vec{\varepsilon}\}$$

The infinite product $\vec{\alpha}$ is defined without any difficulty as the limit of the increasing sequence $\vec{\beta}_n = \vec{\alpha}_1, \ldots, \vec{\alpha}_n$.

But then there exists N_p such that $n \geq N_p$ implies :

$$\pi_i(\vec{\beta}_n^{(j)}) = \pi_i(\vec{\alpha}_j) \text{ for all } j = 1,\ldots,m_p$$

Whence for all $n \geq N_p$, $\pi_i(\vec{\gamma}_n)[p] = \pi_i(\vec{\alpha})[p]$. □

In order to establish the reverse inclusion $\overline{R^\infty} \subseteq \bar{R}^\infty$. We shall make an induction on k. The result is known for k = 1 that is in the case of languages.

<u>Property 15</u> : For all $L \subset A^\infty$ $\overline{L^\infty} = \bar{L}^\infty$.

<u>Proof</u> :

$$(L^\infty)^{fin} = (L^{fin})^*$$
$$(L^\infty)^{inf} = (L^{fin})^* L^{inf} \cup (L^{fin})^\omega$$
$$(\bar{L})^{fin} = L^{fin}$$
$$(\bar{L})^{inf} = Adh(L)$$

From these identities we compute both sides of the equation :

$$(\overline{L^\infty})^{fin} = (L^\infty)^{fin} = (L^{fin})^* \text{ which is equal to :}$$
$$(\bar{L}^\infty)^{fin} = (\bar{L}^{fin})^* = (L^{fin})^*$$
$$(\overline{L^\infty})^{inf} = Adh(L^\infty) = Adh(FG(L^\infty))$$

But $FG(L^\infty) = FG(L^*) = (L^{fin})^* FG(L)$ and we can use formulae established in [ ].

$$Adh(FG(L^\infty)) = (L^{fin})^* Adh(L) \cup (L^{fin})^\omega$$

And this is equal to :

$$(\bar{L}^\infty)^{inf} = (\bar{L}^{fin})^* \bar{L}^{inf} \cup (\bar{L}^{fin})^\omega$$
$$= (L^{fin})^* Adh(L) \cup (L^{fin})^\omega \qquad \square$$

We <u>assume</u> now that $\overline{R^\infty} \subseteq \bar{R}^\infty$ has been proved for all k' < k and consider $\vec{\alpha} = \lim \vec{\alpha}_n$ in R^∞.

A very simple case is when $\vec{\alpha}$ has a finite component :

$$\pi_i(\vec{\alpha}) = f \in A_i^*$$

The sequence $\pi_i(\vec{\alpha}_n)$ is stationary and there exists thus $N \in \mathbb{N}$ such that $n \geq \mathbb{N} \Rightarrow \pi_i(\vec{\alpha}_n) = f$. Equivalently $n \geq \mathbb{N} \Rightarrow \vec{\alpha}_n \in R^\infty(i,f) \times \{f\}$. This implies $\vec{\alpha} \in \overline{R^\infty(i,f) \times \{f\}} = \overline{R^\infty(i,f)} \times \{f\}$

We can write $R^{\infty}(i,f)$ as the finite union of all the products of the form $R(i,\varepsilon)^{\infty}\ R(i,f_1)\ R(i,\varepsilon)^{\infty} \ldots R(i,f_{\ell})\ R(i,\varepsilon)^{\infty}$ where $f_1,\ldots,f_{\ell} \in A_i^+$ are such that $f = f_1,\ldots,f_{\ell}$.

The closure of $R^{\infty}(i,f)$ is the union of the closures of these products that is the union of $\overline{R(i,\varepsilon)^{\infty}}\ \overline{R(i,f_1)} \ldots \overline{R(i,f_{\ell})}\ \overline{R(i,\varepsilon)^{\infty}}$

By induction we know that :

$$\overline{R(i,\varepsilon)^{\infty}} \subseteq \overline{R(i,\varepsilon)}^{\infty} = \bar{R}(i,\varepsilon)^{\infty}$$

whence, since $\overline{R(i,f_j)} = \bar{R}(i,f_j)$.

contained

$\overline{R(i,\varepsilon)^{\infty}}\ \overline{R(i,f_1)} \ldots \overline{R(i,f_{\ell})} - \overline{R(i,\varepsilon)^{\infty}}$ is contained in $\bar{R}(i,\varepsilon)^{\infty}\ \bar{R}(i,f_1)\ldots$ $\bar{R}(i,f_{\ell})\ \bar{R}(i,\varepsilon)^{\infty}$. The last product is obviously contained in $\bar{R}^{\infty}(i,f)$.

Consider now the case $\pi_i(\vec{\alpha})$ is infinite for all $i \in [k]$ where $\vec{\alpha} = \lim \vec{\alpha}_n$.

We can find $\vec{\beta}_o^{(1)}$ and $\vec{\gamma}_o^{(1)}$ such that :

$$\vec{\alpha}[1] \leq \vec{\beta}_o^{(1)},\ \vec{\beta}_o^{(1)} \in \bar{R}^{\infty}\,\bar{R},\ \vec{\gamma}_o^{(1)} \in \overline{R^{\infty}}$$

and :

$$\vec{\alpha} = \vec{\beta}_o^{(1)}\ \vec{\gamma}_o^{(1)}.$$

From $\vec{\alpha}[1] = \vec{\alpha}_n[1]$ for all sufficiently large n's we deduce that there exists a factorization of $\vec{\alpha}_n$ in $\vec{\alpha}_n = \vec{\beta}_n\ \vec{\gamma}_n$, $\vec{\beta}_n \in R^{\ell_n}$, $\vec{\gamma}_n \in R^{\infty}$ such that $\vec{\alpha}_n[1] = \vec{\beta}_n[1]$.

In fact we suppose that ℓ_n is minimum, i.e. we consider $\vec{\alpha}_n = \vec{\beta}_n\ \vec{\gamma}_n$ where $\vec{\beta}_n \in R^{\ell_n}$, $\vec{\gamma}_n \in R^{\infty}$, $\vec{\alpha}_n[1] = \vec{\beta}_n[1]$ and for all factorizations of $\vec{\beta}_n$ in $\vec{\beta}'_n$, $\vec{\beta}''_n$, $\vec{\beta}'_n \in R^{\ell_n - 1}$, $\vec{\beta}''_n \in R$, $\vec{\alpha}_n[1] \neq \vec{\beta}'_n[1]$.

Since α_n converges we can extract convergent subsequences, β_n, $n \in N'$, γ_n, $n \in N'$ which converge towards $\vec{\beta}_o^{(1)}$ and $\vec{\gamma}_o^{(1)}$ such that $\vec{\alpha} = \vec{\beta}_o^{(1)}\ \vec{\gamma}_o^{(1)}$.

Two things may happen ℓ_n, $n \in N'$ is ultimately bounded by M.

Then β_n, $n \in N'$ converges towards $\beta_o^{(1)} \in \overline{R^{\leq M}}$ which is obviously equal to $\bar{R}^{\leq M}$.

We can write $\beta_o^{(1)} \subseteq \bar{R}^{\infty}\,\bar{R}$ since $\vec{\varepsilon} \in \bar{R}^{\infty}$ and certainly since $\beta_o^{(1)}[1] = \vec{\alpha}[1]$, $\beta_o^{(1)} \subset \bar{R}^{\ell}$ for some $\ell \geq 1$. ℓ_n is not ultimately bounded.

For all β_n consider $\vec{\beta}_n = \vec{\beta}'_n\ \vec{\beta}''_n$ with $\vec{\beta}'_n \in R^{\ell_n - 1}$ and $\vec{\beta}''_n \in R$.

We certainly have, since $\vec{\alpha}[1] \neq \beta'_n[1]$, $\pi_i(\beta'_n[1]) = \varepsilon$ for some i which implies :

$$\exists i \; \pi_i(\beta'_n[1]) = \varepsilon \text{ for infinitely many } n\text{'s in } N'.$$

Then $\vec{\beta}'_n$, for these n's in N", belongs to :

$$R^{\ell_n - 1}(i,\varepsilon) \times \varepsilon$$

And we can write the limit of $\vec{\beta}_n = \vec{\beta}'_n \vec{\beta}''_n$ as the product of two converging subsequences of $\vec{\beta}'_n$ and $\vec{\beta}''_n$ i.e. as $\vec{\beta}_o^{(1)'} \vec{\beta}_o^{(1)''}$ where $\vec{\beta}_o^{(1)'} = \lim \vec{\beta}'_n$ belongs to $R^{\infty}(i,\varepsilon)\times\varepsilon$ which is contained by induction in $\overline{R}^{\infty}(i,\varepsilon) \times \{\varepsilon\}$ itself contained in $\overline{R}^{\infty}$. $\vec{\beta}_o^{(1)''} = \lim \vec{\beta}''_n$ belongs to $\overline{R}$ since $\vec{\beta}''_n \in R$. We have proved what was annonced.

Repeating the process we can find a sequence :

$$\vec{\beta}_o^{(1)} \vec{\beta}_o^{(2)} \ldots \vec{\beta}_o^{(n)} \ldots \text{ such that for all } n \in \mathbb{N}_+$$

$$\vec{\alpha}[n] = (\vec{\beta}_o^{(1)} \ldots \vec{\beta}_o^{(n)})[n]$$

The sequence $\vec{\beta}_o^{(1)} \ldots \vec{\beta}_o^{(n)}$ obviously converges towards $\vec{\alpha}$ and we have written $\vec{\alpha}$ as an infinite product $\vec{\alpha} = \vec{\beta}_o^{(1)} \ldots \vec{\beta}_o^{(n)} \ldots$ of elements in $\overline{R}^{\infty}\,\overline{R}$.

To get our result it then suffices to prove the :

<u>Lemma 3</u> : For all R $(R^{\infty})^{\infty} = R^{\infty}$.

<u>Proof</u> : We know that $(R^{\infty})^{fin} = (R^{fin})^{*}$ and $(R^{\infty})^{inf} = (R^{fin})^{*} R^{inf} \cup (R^{fin})^{\omega}$. And we compute :

$$((R^{\infty})^{\infty})^{fin} = ((R^{\infty})^{fin})^{*} = ((R^{fin})^{*})^{*} = (R^{fin})^{*} = (R^{\infty})^{fin}$$

$$\begin{aligned} ((R^{\infty})^{\infty})^{inf} &= ((R^{\infty})^{fin})^{*} (R^{\infty})^{inf} \cup ((R^{\infty})^{fin})^{\omega} \\ &= ((R^{fin})^{*})^{*} [(R^{fin})^{*} R^{inf} \cup (R^{fin})^{\omega}] \cup ((R^{fin})^{*})^{\omega} \\ &= (R^{fin})^{*} R^{inf} \cup (R^{fin})^{\omega} = (R^{\infty})^{inf}. \quad \square \end{aligned}$$

<u>Exemple</u> : The following exemple shows that we cannot really simplify our proof. We take :

$$\alpha_n = \langle a,\varepsilon\rangle^n \langle \varepsilon,b\rangle^n \; ; \; n \in \mathbb{N}_+$$

This sequence obviously converges towards $\langle a^{\omega}, b^{\omega}\rangle = \vec{\alpha}$.

But $\vec{\alpha}_n = \vec{\beta}_n \vec{\gamma}_n$ and $\vec{\alpha}[1] = \langle a,b\rangle \leq \vec{\beta}_n[1]$ implies $\vec{\beta}_n = \langle a,\varepsilon\rangle^n \langle \varepsilon,b\rangle \in R^{n+1}$.

Writing $\vec{\beta}_n = \vec{\beta}'_n \vec{\beta}''_n$ with $\vec{\beta}'_n = \langle a,\varepsilon\rangle^n$, $\vec{\beta}''_n = \langle\varepsilon,b\rangle$ we get :

$$\vec{\beta}_o^{(1)} = \lim \vec{\beta}_n = \lim \vec{\beta}'_n \lim \vec{\beta}''_n = \langle a^\omega,\varepsilon\rangle \langle\varepsilon,b\rangle = \langle a^\omega,b\rangle$$

The factorization of $\vec{\alpha}$ we can obtain from $\vec{\alpha} = \lim \vec{\alpha}_n$ is :

$$\vec{\alpha} = \langle a^\omega,b\rangle \langle a^\omega,b\rangle \ \ldots \ \langle a^\omega,b\rangle \ldots$$

From theorem 1 we can derive a formula to compute the adherence of the star a relation which is exactly the same as the formula to compute the adherence of the star a language.

Corollary 2 : $\mathrm{Adh}(R^*) = (R^{fin})^* \, \mathrm{Adh}(R^\infty) \cup (R^{fin})^\omega$

Proof :

$$\begin{aligned}\mathrm{Adh}(R^*) &= (\overline{R^*})^{inf} = (\overline{R^\infty})^{inf} = (\bar{R}^\infty)^{inf} \\ &= (\bar{R}^{fin})^* \, (\bar{R}^{inf}) \cup (\bar{R}^{fin})^\omega \\ &= (R^{fin})^* \, \mathrm{Adh}(R) \cup (R^{fin})^\omega \qquad \square\end{aligned}$$

5.4. Composition of relations

Let $I+J_1$ and $I+J_2$ be subsets of $[k]$ such that $I+J_1 \cap I+J_2 = \emptyset$.

We can compose over I two relations $R_1 \subseteq \mathcal{R}_{I+J_1}$ and $R_2 \subseteq \mathcal{R}_{I+J_2}$ to get :

$$R_1 \circ_I R_2 = \{\vec{\alpha} \times \vec{\gamma} \mid \vec{\alpha} \in \mathcal{R}_{J_1} \ \vec{\gamma} \in \mathcal{R}_{J_2} \ \exists \vec{\beta} \in \mathcal{R}_I : \vec{\alpha} \times \vec{\beta} \in R_1 \text{ and } \vec{\beta} \times \vec{\gamma} \in R_2\}$$

Example : The following example shows that $\overline{R_1 \circ_I R_2} = \bar{R}_1 \circ \bar{R}_2$ is not usually true.

Take $J_1 = \{1\}$, $I = \{2\}$, $J_2 = \{3\}$:

$$R_1 = \langle a^n, (b^{2n})^p \mid n, p \in \mathbb{N}_+\rangle$$

$$R_2 = \langle b(b^{2n})^p, c^n \mid n, p \in \mathbb{N}_+\rangle$$

The composition over I.

$R_1 \circ_2 R_2$ is clearly empty since $\pi_2(R_1) \cap \pi_2(R_2) = \emptyset$.

But : $\bar{R}_1 = R_1 \cup a^+ \times \{b^\omega\} \cup \langle a^\omega, b^\omega \rangle$

$\bar{R}_2 = R_1 \cup \{b^\omega\} \times c^+ \cup \langle b^\omega, c^\omega \rangle$

And $\bar{R}_1 \circ R_2$ contains $a^+ \times c^+ \cup \langle a^\omega, c^\omega \rangle$. □

Theorem 2 : If $FG(R_1) \subseteq R_1$ and $FG(R_2) \subseteq R_2$ then $\overline{R_1 \circ_I R_2} = \bar{R}_1 \circ_I \bar{R}_2$.

Proof : The inclusion $\overline{R_1 \circ_I R_2} \subseteq \bar{R}_1 \circ \bar{R}_2$ ... is always true.

Every $\vec{\delta} = \vec{\alpha} \times \vec{\gamma} \in \overline{R_1 \circ_I R_2}$ is the limit of a séquence $\vec{\delta}_n = \vec{\alpha}_n \times \vec{\gamma}_n$ where $\vec{\alpha}_n \times \vec{\gamma}_n \in R_1 \circ_I R_2$. Thus for all n there exists $\vec{\beta}_n \in \mathscr{A}_I$ such that :

$$\vec{\alpha}_n \times \vec{\beta}_n \in R_1 \quad \text{and} \quad \vec{\beta}_n \times \vec{\gamma}_n \in R_2$$

We can extract from $\vec{\beta}_n$ a converging subsequence, $\vec{\beta}_n$, $n \in N'$ with $\vec{\beta}$ as a limit.

Clearly the sequences $\vec{\alpha}_n \times \vec{\beta}_n$, $n \in N'$ and $\vec{\beta}_n \times \vec{\gamma}_n$, $n \in N'$ converge towards respectively $\vec{\alpha} \times \vec{\beta}$ and $\vec{\beta} \times \vec{\gamma}$. Since $\vec{\alpha} \times \vec{\beta} \in \bar{R}_1$ and $\vec{\beta} \times \vec{\gamma} \in \bar{R}_2$, we have : $\vec{\delta} = \vec{\alpha} \times \vec{\gamma} \in \bar{R}_1 \circ_I \bar{R}_2$.

Conversely assume $FG(R_i) \subseteq R_i$ for $i = 1,2$. Consider $\vec{\delta} = \vec{\alpha} \times \vec{\gamma} \in \bar{R}_1 \circ_I \bar{R}_2$.

These exists $\vec{\beta}$ such that $\vec{\alpha} \times \vec{\beta} \in \bar{R}_1$ and $\vec{\beta} \times \vec{\gamma} \in \bar{R}_2$. We can write :

$$\vec{\alpha} \times \vec{\beta} = \lim (\vec{\alpha}_n \times \vec{\beta}_n) \text{ where } \forall n \ \ \vec{\alpha}_n \times \vec{\beta}_n \in R_1$$

$$\vec{\beta} \times \vec{\gamma} = \lim (\vec{\beta}'_n \times \vec{\gamma}_n) \text{ where } \forall n \ \ \vec{\beta}'_n \times \vec{\gamma}_n \in R_2$$

Then for all $p \in \mathbb{N}$ there exists N such that :

$$n \geq N \Rightarrow \vec{\alpha}_n[p] = \vec{\alpha}[p],\ \vec{\gamma}_n[p] = \vec{\gamma}[p]$$

$$\text{and} \qquad \vec{\beta}_n[p] = \vec{\beta}[p] = \vec{\beta}'_n[p].$$

Whence $\vec{\alpha}[p] \times \vec{\beta}[p] = \vec{\alpha}_n[p] \times \vec{\beta}_n[p] = (\vec{\alpha}_n \times \vec{\beta}_n)[p]$ is in $FG(R_1) \subseteq R_1$ and similarly $\vec{\beta}[p] \times \vec{\gamma}[p]$ is in R_2.

The sequence $\vec{\alpha}[p] \times \vec{\gamma}[p]$, $p \in \mathbb{N}$ obviously converges towards $\vec{\delta} = \vec{\alpha} \times \vec{\gamma}$ and is composed of elements of $R_1 \circ_I R_2$, where $\vec{\delta} \in \overline{R_1 \circ_I R_2}$. □

Corollary : For all relations R_1, R_2

$$Adh(R_1) \circ Adh(R_2) \cup Adh(R_1) \circ FG(R_2) \cup FG(R_1) \circ Adh(R_2) = Adh(FG(R_1) \circ FG(R_2)).$$

Proof :

$$\begin{aligned}\mathrm{Adh}(FG(R_1) \circ FG(R_2)) &= (\overline{FG(R_1) \circ FG(R_2)})^{\mathrm{inf}} \\ &= (\overline{FG(R_1)} \circ \overline{FG(R_2)})^{\mathrm{inf}} \\ &= ((\bar{R}_1 \cup FG(R_1)) \circ (\bar{R}_2 \cup FG(R_2)))^{\mathrm{inf}} \\ &= (\bar{R}_1)^{\mathrm{inf}} \circ (\bar{R}_2 \cup FG(R_2)) \cup (\bar{R}_1 \cup FG(R_1))(\bar{R}_2)^{\mathrm{inf}} \quad \square\end{aligned}$$

CONCLUSION

We have obtained a number of results concerning the topological closure of infinitary relations : in practice, at least for modeling the synchronization of concurrent processes, we shall use mainly infinitary rational relations. A forthcomming paper of the same author is devoted to theim definition and properties. The author has had very helpful discussions with A. Arnold, L. Boasson, F. Boussinot, G. Roncairol and G. Ruggin.

BIBLIOGRAPHY

[1] A. ARNOLD and M. NIVAT
Metric interpretations of infinite trees and semantics of non deterministic recursive programs. Theor. Comp. Sci., Vol. 11 (1980), 181-205.

[2] J. BEAUQUIER and M. NIVAT
Application of formal language theory to problems of security and synchronization, in Formal Language Theory (R. Book, éd.) Academic Press, New York, 1980.

[3] L. BOASSON and M. NIVAT
Adherences of languages, Jour. Comp. Syst. Sci., Vol. 20 (1980), 285-309.

[4] S. EILENBERG
Automata, Languages and Machines, Vol. A, Academic Press, New York, 1974.

[5] M. NIVAT
Systèmes de transition permanents et équitables, Research Report n° 2577, Laboratoire Central de Recherches Thomson-CSF, Orsay, 1980.

[6] M. NIVAT
Infinitary languages (to appear).

[7] M. NIVAT
Synchronization et multimorphismes (to appear).